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Faculty of Mathematics and Physics

Polyhedra and Secondary School Mathematics

Michaela Koblížková

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Michaela Koblížková

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Doktorand:	Michaela Koblížková
Školitel:	Doc. RNDr. Jiří Kadleček, CSc.,
Školící pracoviště:	Katedra didaktiky matematiky MFF UK
Oponenti:	Prof. RNDr. František Kuřina, CSc., Katedra matematiky Pedagogická fakulta UHK Víta Nejdedlého 573, Hradec Králové, 500 03
	Doc. RNDr. Leo Boček, CSc., Katedra didaktiky matematiky Matematicko-fyzikální fakulta UK Sokolovská 83, Praha 8, 186 75

Autoreferát byl rozeslán dne: ....

Obhajoba se koná dne .... před komisí pro obhajoby dizertačních prací v oboru m-8 Obecné otázky matematiky a informatiky na Matematicko-fyzikální fakultě Univerzity Karlovy, Ke Karlovu 3, Praha 2, v místnosti č. 105.

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Předseda oborové rady: Doc. RNDr. Jindřich Bečvář, CSc.,
Matematický ústav UK,
Sokolovská 83, Praha 8, 186 75

Introduction

The thesis consists of two independent parts:

In the first part reasons are given for general conception, destination and methodic approach to the latter part, which is the book ***Let Us Get Insight into Polyhedra***, a collection of solved problems from the field of solid geometry. Each part of the thesis is fully independent. It has its own contents, bibliography and paging.

The book connects secondary-school level of mathematical knowledge with tasks of Mathematical Olympics. It is meant for self-studying. The author has done her best to reach the largest possible extent of problems included.

Every chapter of the book consists of one or two pages introducing problems and several pages of their solutions. The main achievement is always repeated and precised in the end of the chapter where *To remember* and *Notes* are usually attached.

1 Process of writing the book and destination of this work

The author wanted:

- to keep way from known to unknown, from particular to abstract,
- to write plainly,
- to present an attractive part of geometry and to practise some other parts of mathematics.

The author wrote her book in three phases:

- At first she was searching for “series of problems” about polyhedra.
- Then she tried to optimalize the number of problems to use more than ones the most interesting methods of solving and to practise a large part of mathematical knowledge.
- Lastly, she struggled for exactness and intelligibility.

Aims followed in the course of writing the book were:

1. to show the reader different ways of transforming space into a plane,
2. to prepare tasks for next chapters,
3. to show qualities of every mentioned body,

4. to connect new information with secondary school knowledge,
5. to practise numerical skill,
6. to show extent and variety of the theme,
7. to use various methods of solving problems,
8. to use various types of proofs,
9. to use systematic classification,
10. to get readers acquinted with some of university knowledge.

Kuřina in [4] says: "*The mathematics means solving problems*", and the types of problems are calculation, decision, recognition, construction and proof.

Geometry as a special part of maths is solving problems as well. That is why the author has created the collection of problems with their solutions to make the reader acquinted with polyhedra. In the book, some motivating or instructive problems in solid geometry are also shown which are, at the same time, good exercise for other parts of mathematics.

2 Methods of solving mathematic problems

Quatations from literature are being referred in this chapter, especially Loren C. Larsons's classification of methods solving mathematical problems, introduced in his book *Problem-Solving Through Problems* [6].

3 Theme of polyhedra

The reasons why the polyhedra was chosen as the topic of the book was not only that it is a very attractive part of geometry but also because this topic has not been worked out for secondary school students yet, except a small booklet Polyhedra by S. Horák in the edition School for Young Mathematicians [2].

The sources of knowledge searched for in the book were *Polyhedra* by R. Cromwell [1] and *Convex Polyhedra* by E. Jucovič [2]. A great motivation for the author was the book *Geometrical Rhapsody* by K. Levitin [7].

4 Main problems

- How to answer the question *What is a polyhedron?* - because "*At different times, to different people, the word* polyhedron *has conjured up a wide variety of images, some of which are incompatible with each other*" - Cromwell in [1].
- To what extend to precise new notions and how to introduce the theorems the proof of which is too difficult for secondary school, eg. Euler's theorem.
- Usage of Czech terminology.
- Varied level of reader's knowledge of descriptive geometry.

5 Summary

Geometry is not only attractive but also very useful for practising other mathematical skills. The book can introduce polyhedra to a reader who can also practise a great deal of his mathematical knowledge at the same time.

A didactic-constructive approach to the subject is used in the book. One of its characteristic features is using various ways of cultivating the mathematical world and its presentation. Partial items of experience and knowledge are organized, classified and ranged in various ways (by Kuřina in [5]).

The way of compiling the book is really the object one as there are more than one hundred drawings in the book.

6 Conclusion

The author took care of talented students for twenty years. She prepared and run 20 cycles of South Bohemia Corresponding Seminar in Mathematics (Pedagogical faculty of South Bohemian University, 1989-2002), both tasks and model solutions. That is why she is able to recognize her book's benefit for gifted students and their teachers.

7 Contents of the book *Let Us Get Insight into Polyhedra* and samples from it

Introduction

7.1 Do you really know the cube?

The first chapter deals with cubes. It also shows transformation of a cube into a plane and "reading" drawings. At the end of the chapter a regular tetrahedron and a regular octahedron is shown as a part of a cube.

The chapter has ten figures and offers ten problems to solve. There are some of them:

Problem 1.5
The plane $\alpha =\leftrightarrow BGD$ cuts the cube $ABCDEFGH$ of the edge length a (Figure 1).

a. What does the cut look like?
b. Count the area of the cut.

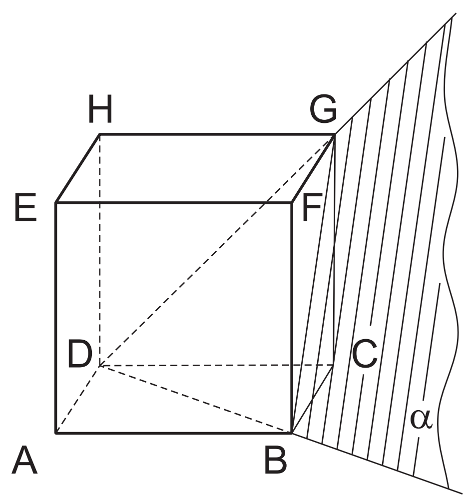

Figure 1: Ad problem 1.5

Problem 1.8
In Figure 1 mark off the plane τ (using its cut) going through the centre of the cube parallel to the plane α.
Solution: (See Figure 2)
The cut is a regular hexagon of the side length c when $c = \frac{\sqrt{2}}{2}a$.

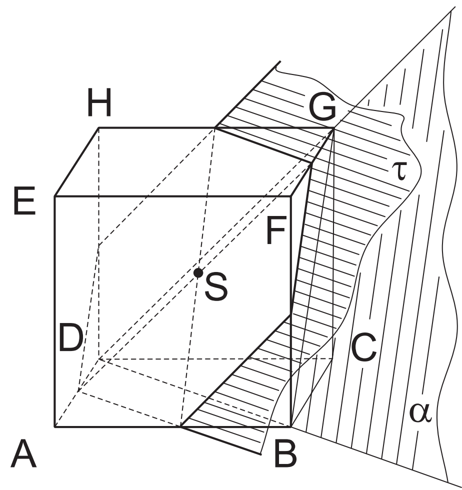

Figure 2: Solution to problem 1.8

Problem 1.9
Complete statements:

a. The solid (polyhedron) $ACFH$ inscribed in the cube $ABCDEFGH$ has only .... faces which are all ....
b. The solid (polyhedron) whose vertices are only the centres of all faces of the given cube $ABCDEFGH$ has only .... faces which are all ....

Notes at the end of the chapter:

1. It is important to be able to observe every side of the situation and to change a direction by requirement.

2. It is necessary to distinguish carefully, which abscissa keeps its real length in a drawing. It applies similarly to the size of a mapped angle.

7.2 Simple and perfect body

The second chapter gets the reader acquainted with characters of regular tetrahedra. The chapter contains eight problems and their solutions, including six figures.

Problem 2.5
Find all square cuts of a regular tetrahedron.

Problem 2.7
We take up:

a. centres of faces of a regular tetrahedron,
b. centres of edges of a regular tetrahedron,

what polyhedron has only these as its vertices?

7.3 Pyramids, prisms, polyhedra

The chapter deals with pyramids, prisms and then with polyhedra generally. *Polyhedron is such a solid, boarder of which is a union of polygons. The polygons are called faces. Every side of each face is also a side of another (neighbouring) face and we call it edge. Neighbouring faces do not lie in the same plane.*

The chapter has five parts, each of them with several problems and figures.

7.3.1 Tetrahedron

Problem 3.1.2
Prove that all absciccae connecting centres of opposite edges of any tetrahedron have common centre.

Problem 3.1.9
Show that the centre of gravity of a tetrahedron is also the centre of every abscissa connecting the centres of opposite edges of the tetrahedron.
Solution:
For the centre O of the abscissa connecting the centres of edges AB and CD is:
$O = [(A+B):2+(C+D):2]:2 = (A+B+C+D):4$.

7.3.2 Pyramids are regular or irregular

7.3.3 Cube is one of prisms

Problem 3.3.6
Show that every parallelepiped has a centre of symmetry.

Problem 3.3.7
Let $ABCDEFGH$ be a parallelepiped. Prove that the inscribed tetrahedra $ACFH$ and $GEDB$ are congruent.
Solution:
Tetrahedra are mutually symmetrical by the centre of symmetry of the parallelepiped to which they are inscribed. (This centre is also their common centre of gravity).

7.3.4 Parallelepiped circumscribed to a tetrahedron

Problem 3.4.3.
Find axes of symmetry of a regular tetrahedron.

7.3.5 What are polyhedra

Problem 3.5.1
Prove the statement: $3V \leq 2E$, where V is the number of all vertices and E is the number of all edges is true for every polyhedron.
Solution:
Each vertex coincides with at least three edges, and every edge has two vertices. That is why $3V \leq 2E$.

Notes at the end of the chapter:

1. It is often useful to complete the tetrahedron to a parallelepiped when we are solving problems about tetrahedra.

3. When two pairs of the opposite edges of a tetrahedron are mutually perpendicular then the same applies for the third pair.

7.4 Euler's theorem and types of polyhedra

The fourth chapter deals with Euler's formula: $F + V = E + 2$, where F is number of faces, V number of vertices and E number of edges. The chapter also mentions Steinitz's theorem about existence of polyhedron to given F, V, E, types of convex polyhedra and duality of convex polyhedra.

The chapter includes two parts, the first one with seven and the latter one with five problems, and ten figures altogether.

7.4.1 Euler's theorem

Problem 4.1.6.
Resolve:

a. What is minimal number E_m of edges of any convex polyhedron?
b. Is there any convex polyhedron with $E_m + 1$ edges?

Solution:

a. For a convex polyhedron with number of faces F, number of vertices V and number of edges E is $F + V = E + 2$, $3V \leq 2E$, $3F \leq 2E$.

then $3F + 3V = 3E + 6 \leq 4E$, $6 \leq E$.
Every tetrahedron has 6 edges.
Conclusion: Minimal number of edges of a polyhedron is $F_m = 6$.

b. If there is a polyhedron with seven edges, it means $F+V = 9$, $3V \leq 14$ and $3F \leq 14$. Then $3V \leq 12$, $3F \leq 12$ and $3F + 3V = 27 \leq 24$ - the polyhedron cannot exist.

Problem 4.1.7
Such a polyhedron, that Euler's teorem is true for, is called *Euler's polyhedron*. Show that we get new Euler's polyhedron by cutting off a vertex of Euler's polyhedron.

7.4.2 Types of polyhedra, duality of polyhedra

Problem 4.2.2
Show that *autoduality* of polyhedron exists. (The meaning of that is: a polyhedron dual to itself exists.)
Solution:
Every tetrahedra (or pyramid) is dual to itself.

Problem 4.2.3
Find polyhedra which are in duality with prisms.

Notes at the end of the chapter:

5. Searching for statements dual to known theorems is a good way how to expand the knowledge.

7.5 Regular tetrahedron and cube have only three more members of the family

The fifth chapter introduces regular polyhedra, various situations where we can meet them, their duality, their nets etc. The chapter has two parts, thirteen problems and ten figures.

7.5.1 Regular polyhedra

Problem 5.1.1
Given tablet shows all types of regular polyhedra and their qualities.

Name of polyhedron	Face is	Number of faces	vertices	edges
Regular tetrahedron	equilateral triangle	4	4	6
Regular hexahedron (cube)	square	6	8	12
Regular octahedron	equilaterral triangle	8	6	12
Regular dodecahedron	regular pentagon	12	20	30
Regular icosahedron	equilateral triangle	20	12	30

Without using Euler's theorem give reason for the statement that there cannot be more of them.
Solution:
In every vertex of a regular polyhedron the same number of faces meet, which are at least three. Summa of their angles by this vertex is less than four right angles. It can be realized only in this way: 3x60°, 4x60°, 5x60°, 3x90° or 3x108°, which corresponds with the five types of bodies in the tablet.

7.5.2 Properties of regular polyhedra

Problem 5.2.9
Imagine, you have bricks in the shape of regular polyhedra:

a. cubes,
b. tetrahedra,
c. octahedra,
d. icosahedra.

What kind of bricks can we use if we need to fill in the space without gaps.

Notes at the end of the chapter:

2. We cannot fill in the space (without gaps) with only one kind of regular polyhedra without cubes. But we can fill it in with both regular tetrahedra and regular octahedra.

7.6 Another polyhedra with regular faces

The sixth chapter shows several new types of bodies, especially Archimedean solids, antiprisms, deltahedra or dipyramids. There are three subchapters with twenty problems and eleven figures.

7.6.1 Convex polyhedra with congruent regular faces

Problem 6.1.3
Make the decision if there is - apart from a cube - another 6-hedron with congruent regular faces.
Solution:
Yes, it is a special dipyramid, which faces are equilateral triangles.

7.6.2 Half-regular polyhedra

Problem 6.2.4
Show that half-regular F-hedron exists for every $F \geq 7$.

Problem 6.2.5
Find at least one half-regular polyhedron such that every of its faces meeting in one vertex has different number of vertices.

7.6.3 Non-convex polyhedra with regular faces

To remember at the end of the chapter:

1. The only convex polyhedron all faces of which are squares is the cube and the only convex polyhedron all faces of which are regular pentagons is the regular dodecahedron.

7.7 Polyhedra sharing given property

Themes of the seventh chapter are: special tetrahedra in the first part, polyhedra with triangular faces and polyhedra with three-valent vertices in the latter part. In this chapter with twenty problems there are only four figures.

7.7.1 Special tetrahedra

Tetrahedra with congruent faces
Tetrahedra inscribed to cuboid or to rhombohedron
Tetrahedra having point of intersection of lines going through vertex perpendicularly to the opposite face
Tetrahedra with rectangular faces

7.7.2 Polyhedra with triangular faces and polyhedra with three-valent vertices

Problem 7.2.7
What type is a polyhedron which is dual to a polyhedron with

a. triangular faces,
b. three-valent vertices?

Solution:

a. A dual polyhedron to a polyhedron with triangular faces is the polyhedron with three-valent vertices.
b. A dual polyhedron to a polyhedron with three-valent vertices is the polyhedron with triangular faces.

Problem 7.2.8
Using Euler's theorem find formulas for numbers of vertices, faces and edges of a polyhedron with

a. triangular faces,
b. three-valent vertices.

Notes at the end of the chapter:

1. All faces of a tetrahedron can be rectangular triangles only when the right angles create two pairs of them having common vertex.

2. The only type of polyhedron which has both triangular faces and three-valent vertices is tetrahedron.

7.8 Cuts and intersections

The chapter deals with plane cuts of polyhedra and intersections of two polyhedra. It offers seventeen problems in three subchapters and ten figures.

7.8.1 Cuts

7.8.2 Cutting of regular polyhedra

Problem 8.2.6
Make a decision if we can find a cut which is a regular hexagon on

a. a cube,
b. a regular octahedron,
c. a regular dodecahedron.

Solution:
We can find a cut which is a regular hexagon on each of mentioned polyhedra. See Figures 2, 3 and 4.

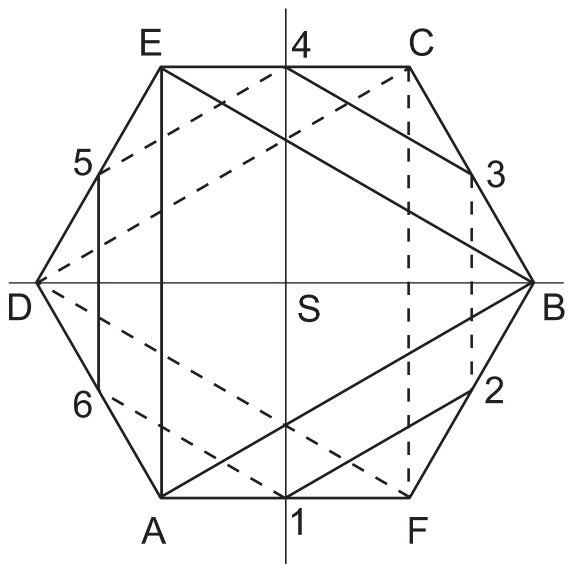

Figure 3: Regular hexagon as a cut of regular octahedron

7.8.3 Intersections

Problem 8.3.1
Regular tetrahedra $ACFH$ and $BDEG$ are inscribed in the same cube $ABCDEFGH$. What is an intersection of both tetrahedra?
Solution:
It is a regular octahedron. See Figure 5.

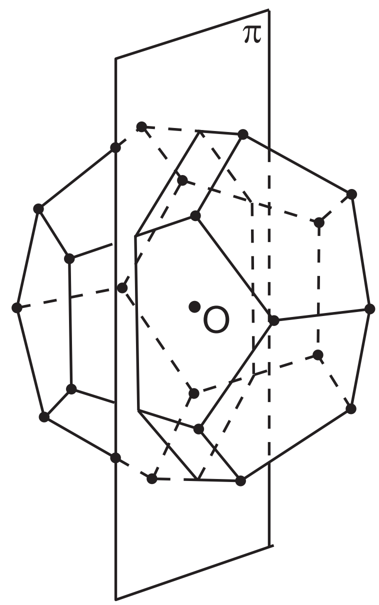

Figure 4: Regular hexagon as a cut of regular dodecahedron

7.9 Nets of polyhedra and how we can use them

The ninth chapter is about nets of polyhedra and about the shortest way on the surface of a polyhedron. This chapter has fifteen problems divided into three subchapters, accompanied by twenty-one figures.

7.9.1 Nets of pyramids

7.9.2 Nets of other polyhedra

Problem 9.2.3
Let $ABCDEF$ be a regular octahedron the edge length of which is a and E, F are its opposite vertices. We cut off the vertices A, B, C, D as if we wanted to create a cuboctahedron (vertices E, F are not cut off). We obtain a dodecahedron. Four of the faces of the dodecahedron are the square cuts just made. Find what kind of polygons the other faces are, and draw the net of the dodecahedron.

Problem 9.2.4
We wish to replace the regular octahedron from the last problem by dipyramid $ABCDVW$ to obtain, as a result of the described cutting, a dodecahedron all faces of which are congruent rhombi. Draw the net of the obtained solid.

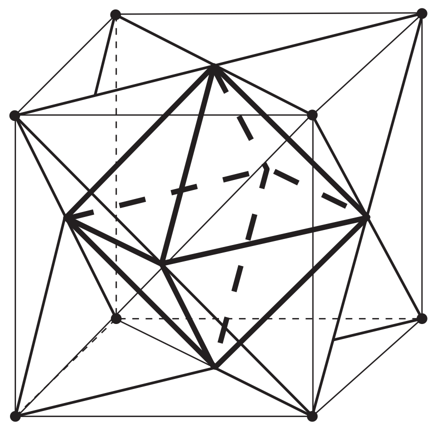

Figure 5: Regular octahedron as an intersection of two special regular tetrahedra

7.9.3 Shortest ways on the surface of a polyhedron

7.10 Sorting and describing convex polyhedra

Types of polyhedra and their diagrams are the topic of the tenth chapter. In two subchapters there are eight problems with eight figures.

7.10.1 Types of convex polyhedra

Problem 10.1.4
Two of the Archimedean solids have an equal number of faces $F = 32$, number of vertices $V = 60$ and number of edges $E = 90$. But they are not of the same type. The first of them has characterics $(12_5, 20_6)$, the other $(20_3, 12_{10})$. What solids are we speaking about and what is the meaning of characteristics in brackets?
Solution:
The polyhedra which we are looking for are a truncated icosahedron and a truncated dodecahedron. The meanings say: The first solid has 12 faces which are 5-gons and 20 faces which are 6-gons, the latter one has 20 triangles and 12 10-gons.

7.10.2 Diagrams of convex polyhedra

Problem 10.2.4
Find at least one convex polyhedron of every type given by the diagrams in Figure 6.

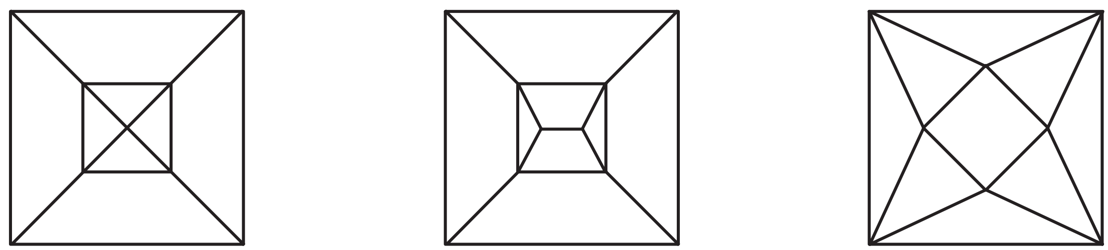

Figure 6: Diagrams of polyhedra

7.11 Rotation and colouring of polyhedra

The eleventh chapter shows rotations of polyhedra, and colourings of polyhedra. There are groups of symmetries of the polyhedron and proper colourings of a given polyhedron in the centre of attention.

"Proper colouring of a polyhedron is such a colouring when faces that share a common edge must have different colours" [1].

This chapter has two parts and fourteen problems with only one picture.

7.11.1 Groups of rotations of polyhedra

7.11.2 Colouring

Problem 11.2.1
We wish to colour faces of a polyhedron so that neighbouring faces must have different colours. Show that there is a polyhedron for whose colouring two colours are enough. How many polyhedra of such a property exist?

Problem 11.2.4
We must colour faces of a regular tetrahedron by four (given) different colours. How many distinguishable results are there?

Notes at the end of the chapter:

2. Proved: In the case of Platonic solid, the number of rotational symmetries equals twice its number of edges.

7.12 Further problems

Finding polyhedra sharing the same properties is the theme of the first subchapter. The latter one shows four problems of various subjects. Each part of the chapter has four problems and there are seven figures altogether.

7.12.1 The set of polyhedra sharing the same properties

Problem 12.1.1
The cube $ABCDEFGH$ is given. Specify the set of the centres of the abscissae XY, with X lying on AC and Y lying on FH.
Solution:
The set is a square cut of tetrahedron $ACFH$.

7.12.2 Various problems

7.13 Historical and other matters of interest

The last chapter mentions historical contexts and other various matters of interest. Descartes' theorem is involved that:
"The sum of deficiencies of the solid angles in polyhedron is eight right angles" [1].

The number of problems is five, with two figures.

Problem 13.2
Find out what kind of polygons a surface of a football is sown of. What is the name of the corresponding polyhedron?
Solution:
A football consists of twelve regular 5-gons and twenty regular 6-gons. The corresponding polyhedron is a truncated icosahedron.

Conclusion

Polyhedra are a fascinating part of geometry.

References

[1] Cromwell, P. R. *Polyhedra*. Cambridge University Press, 1997.

[2] Horák, S. *Mnohostěny (Polyhedra)*. Praha: ÚV matematické olympiády, Škola mladých matematiků N. 27, 1970.

[3] Jucovič, E. *Konvexné mnohosteny (Convex polyhedra)*. Bratislava: Veda, 1981.

[4] Kuřina, F. *Matematika je řešení úloh (Mathematics means solving problems)* in Matematika-fyzika-informatika 2003, N. 3, p. 129-142.

[5] Kuřina, F. *Transformační pojetí školské geometrie a konstrukční přístupy k vyučování matematice (The transformatic conception of the school geometry and constructive approach to teaching mathematics)* in Pokroky matematiky, fyziky a astronomie 1999, N. 1, p. 75-83.

[6] Larson, L. C. *Metódy riešenia matematických problémov (Problem-Solving Through Problems)*. Bratislava: Alfa, 1990.

[7] Levitin, K. *Geometrická rapsódie (Geometrical Rhapsody)*. Praha: Státní nakladatelství technické literatury, 1991.